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A Review for the Time Integration of Semi-Linear Stiff Problems

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ABSTRACT

Several real-world requests that involve conditions where different physical phenomena perform on very different time scales arise simultaneously. The partial differential equations (PDEs) that manage such situations are classified as stiff PDEs. Stiffness is a difficult property of differential equations (DEs) that avoid conservative explicit numerical integrators from managing problem efficiency. There has also been a large compact of importance in the building of exponential integrators. However, different some of the new literature proposes, integrators based on this philosophy have been confirmed since at least 1960.The aim of this study is to review the time integration proposed for semi-linear stiff problems.
KEYWORDS: Exponential methods, numerical inverse Laplace transform, semi-linear parabolic equation.

1. INTRODUCTION

The theory of numerical methods for the time integration of semi-linear stiff problems is well-proposed by the application of exponential methods. Cox and Matthews (2002) studied a clear derivation of the explicit Exact Linear Part (ELP) method, from which they referred the methods as Exponential Time Differencing (ETD) and their implementation of the ETD methods. A modification of the ETD Runge-Kutta schemes of Cox and Matthews has been claimed by Kassman and Terfethen (2005). Maria Lopez-Fernandez (2004) discussed a new algorithm for the implementation of exponential methods ,and the algorithm evaluates the operator by the exponential methods with a quadrature formula that is converges. A spectral order method for inverting sectorial Laplace transforms was studied by Maria Lopez-Fernandez et al. (2006). Moreover, the class of explicit multistep exponential and the explicit exponential Runge-Kutta methods were discussed by Hochbruck and Ostermann (2005).A representation for operators required in the implementation of these integrators in term of suitable Laplace transforms was proposed by Lopez-Fernandez et al. (2005). Other papers on this subject include [1, 4, 5, 7, 10,11,12,13, 14,15,16,17,18,19, 20,21,22, 23, 24, 25].

The problems under consideration can be written as follows:

$$u'(t) = A\,u(t) + f(t, u(t)) \qquad u(0) = u_0 \qquad 0 \le t \le T, \tag{1}$$

where$A$ is a linear operator that represents the highest order of differential terms and $f$is a lower under nonlinear operator. The variation of constants formula is considered for the solution to the initial value problem (1).

In Section 2, multistep exponential methods are discussed. In Section 3, exponential Runge-Kutta methods are proposed. In Section 4,evaluationof the mapping required is constructed by the multistep and the Runge-Kutta methods. In Section 5, numerical illustration is presented. Finally, the result is presented.

2. MULTISTEP EXPONENTIAL METHODS

The time integration of problems (1) is demonstrated by a class of explicit exponential methods [16]. with$A : D(A) \subset X \to X$ the infinitesimal generator of a C_0-semi group e^{tA}, $t \ge 0$, of linear bounded operators in a Banach space X [7]. The case of A in (1) is sectorial, i.e., A is a densely defined and closed linear operator on X and there existsconstants $M > 0$, $\gamma \epsilon R$, and an angle $0 < \delta < \frac{\pi}{2}$, such that the resolvent fulfils

$$\|(zI - A)^{-1}\| \le \frac{M}{|z-\gamma|} \quad \text{for } |\arg(z-\gamma| < \pi - \delta. \tag{2}$$

Then, for $0 \le \alpha < 1$, the fractional powers $(\omega - A)^{\alpha}$ are defined for $\omega > \gamma$, and $X_\alpha = D((\omega - A)^{\alpha})$ endowed with the graph norm $\|.\|_\alpha$ is a Banach space [7]. The nonlinearity f in (1) is supposed to be defined on $[0,T] \times X_\alpha \to X$, for some $0 \le \alpha < 1$, and to be locally Lipschitz in a strip along the exact solution. Therefore, there exists $L(R,T) > 0$ such that

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$$\|f(t,\gamma) - f(t,\xi)\| \leq L\|\gamma - \xi\|_\alpha \quad \gamma,\xi \in X_\alpha, \quad 0 \leq t \leq T, \tag{3}$$

for $\max(\|\gamma - u(t)\|_\alpha, \|\xi - u(t)\|_\alpha) \leq R$.

The variation of constants formula in interval $[t_n, t_{n+k}]$ is presented for k −step method $(k \geq 1)$.

Setting a step size $h = {}^{T}/_{N}$, $N \geq k$, and the corresponding time levels $t_n = nh$,

$0 \leq n \leq N$, the solution (1) at t_{n+k} is given by

$$u(t_{n+k}) = u(t_n)e^{khA} + h\int_0^k e^{(k-\sigma)hA} f\big(t_n + \sigma h, u(t_n + \sigma h)\big)d\sigma \quad . \tag{4}$$

Given approximations$u_{n+j} \approx u(t_{n+j})$, $0 \leq j \leq k-1$, after replacing f in (4) by the Lagrange interpolation polynomial of degree $k-1$, $P_{n,k-1}$ through the points $\{(t_{n+j}, f(t_{n+j}, u_{n+j}))\}_{j=0}^{k-1}$ and integrating, the approximation$u_{n+k} \approx u(t_{n+k})$ is obtained.

$$P_{n,k-1}(t_n + \sigma h) = \sum_{j=0}^{k-1}\binom{\sigma}{j}\Delta^j f_n, \tag{5}$$

with $f_m = f(t_m, u_m)$, $0 \leq m \leq N-1$,and Δ the standard forward difference from

$$u_{n+k} = \varnothing_0(k,hA)\,u_n + h\sum_{j=0}^{k-1}\varnothing_{j+1}(k,hA)\,\Delta^j f_n, \tag{6}$$

where

for $\in \mathbb{C}$, $k \geq 1$, $0 \leq j < k$, and $\varnothing_j(k,\lambda)$ are given by

$$\varnothing_j(k,\lambda) = \int_0^k e^{(k-\sigma)\lambda}\binom{\sigma}{j-1}\,d\sigma, \qquad 1 \leq j \leq k$$

$$\varnothing_0(k,\lambda) = e^{k\lambda}, \tag{7}$$

The methods considered in (6) are explicit and they require the evaluation$\varphi_j(k,hA)$.

3. EXPONENTIAL RUNGE-KUTTA METHODS

In the time integration of semi-linear parabolic problems, explicit exponential Runge-Kutta methods have been demonstrated [6, 10].

For $h = {}^{T}/_{N}$,$N \geq 1$,and $1 \leq i \leq s$, the approximation u_n to $u(t_n)$,with $t_n = nh$, are presented by

$$U_{ni} = u_n e^{c_i hA} + h\sum_{j=1}^{i-1} a_{ij}(hA) f(t_n + c_j h, U_{nj}),$$

$$u_{n+1} = u_n e^{hA} + h\sum_{i=1}^{s} b_i(hA) f(t_n + c_i h, U_{ni}), \tag{8}$$

with $c_1 = 0(U_{n1} = u_n)$.

The coefficients $b_i(\lambda)$ and $a_{ij}(\lambda)$ are linear combinations of$\varphi_k(\lambda)$ and $\varphi_k(c_l\lambda)$ with

$$\varphi_k(\lambda) = \int_0^1 e^{(1-\sigma)\lambda}\frac{\sigma^{k-1}}{(k-1)!}\,d\sigma, \qquad \lambda \in \mathbb{C}, k \geq 1, t > 0. \tag{9}$$

Take$\varphi_0(\lambda) = e^\lambda$. The implementing of (8) requires the evaluation of $\varphi_k(hA)$ and$\varphi_k(c_l hA)$,for$1 \leq l \leq s$ and several values of $k \geq 0$. Besides, the nonlinearity f in (1) has satisfied a local Lipchitz condition [16].

4. Evaluation of the vector-valued mapping

Some Laplace transformation formulas are used for getting a suitable representation of the operators $\varnothing_j(k,hA)$, $\varphi_j(hA)$, and $\varphi_j(c_l hA)$ applied in (6) and (8). For a locally integral mapping $f:(0,\infty) \to X$, bounded by

$$\|f(t)\| \leq Ct^{v-1}e^{\gamma t}, \qquad \text{for some } \gamma, v > 0.$$

For some γ,$v > 0$, the Laplace transform is noted by

$$F(z) = L[f](z) = \int_0^\infty e^{-tz} f(t)\,dt, \qquad Re(z) > \gamma .$$

The inverse transform is achieved by means of a suitable rule to discretize the inversion formula

$$f(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt}F(z)\,dz,$$

whereΓ is a contour in the complex plan, running from $-i\infty$ to $i\infty$ and laying in the analytical region of F, and it is noted that $f(\sigma) = L^{-1}[F](\sigma)$ [8, 9, 11, 12, 13].

4.1 Evaluation of the mapping required by the multistep methods

For $\varnothing_j$ in (7) with $1 \leq j \leq k$, it is considered [8, 9, 15]

$$\emptyset_j(k,\lambda) = \int_0^k e^{(k-\sigma)\lambda} \binom{\sigma}{j-1}\, d\sigma = L^{-1}\left[L[f_0(.,\lambda)] \times L[f_j] \right](k)\,,$$

where, for $\upsilon > 0$,

$$f_0(\upsilon,\lambda) = e^{\sigma\lambda} \text{and} f_j(\sigma) = \binom{\sigma}{j-1}. \qquad (10)$$

For every $j \geq 1$and $z \in \mathbb{C}$, the following formula is defined as

$$\Psi_j(z,\lambda) = L[f_0(.,\lambda)](z) \times L[f_j](z) = \frac{1}{z-\lambda} \times L[f_j](z)\,. \qquad (11)$$

Thus, for every $\lambda \in \mathbb{C}$and $j \geq 1$,

$$\emptyset_j(k,\lambda) = L^{-1}[\Psi_j(z,\lambda)](k)\,. \qquad (12)$$

For $j = 0$

$$\emptyset_0(k,\lambda) = e^{\upsilon\lambda} = L^{-1}\left(\frac{1}{-\lambda}\right)(k),$$

and thus the following formula is defined

$$\Psi_0(z,\lambda) = \frac{1}{z-\lambda}. \qquad (13)$$

For λ scalar, the mappings$\Psi_j(z,\lambda)$, with $1 < j < 4$ are presented by

$$\Psi_1(z,\lambda) = \frac{1}{z(z-\lambda)}, \qquad \Psi_2(z,\lambda) = \frac{1}{z^2(z-\lambda)},$$
$$\Psi_3(z,\lambda) = \frac{2-z}{2z^3(z-\lambda)}, \qquad \Psi_4(z,\lambda) = \frac{3-3z-z^2}{3z^4(z-\lambda)}. \qquad (14)$$

In order to evaluate $\emptyset_j(k,hA)$, $1 \leq j \leq 4$, the formulas (14) with hA instead of λ are proposed.For approximating the original mapping at $\sigma = k$, the inversion of the Laplace transform has been performed [8, 9]. In this way, the following formulas are invert of Laplace transforms that we need:

$$\Psi_0(z,hA) = (zI - hA)^{-1},$$
$$\Psi_1(z,hA) = \frac{1}{z}(zI - hA)^{-1},$$
$$\Psi_2(z,hA) = \frac{1}{z^2}(zI - hA)^{-1}, \qquad (15)$$
$$\Psi_3(z,hA) = \frac{2-z}{2z^3}(zI - hA)^{-1},$$
$$\Psi_4(z,hA) = \frac{3-3z+z^2}{3z^4}(zI - hA)^{-1}.$$

Note that because of (2), the mapping $\Psi(z,hA)$ turns out to be sectorial in the variable z, i.e., there exists constants $\gamma \epsilon R$ and $M > 0$, possibly different form of the constants in (2), such that $\Psi(z,hA)$is analytic for z in the sector $|\arg\,(z-\gamma| < \pi - \delta$ and there

$$\|\Psi(z,hA)\| \leq \frac{M}{|z-\gamma|^{v}}\,, \qquad \text{for some } v \geq 1. \qquad (16)$$

The formulas in (15) can also be considered by combining the Cauchy integral formula [7, 8, 14] with the inversion formula for Laplace transform. For suitable contours Γ_1 and Γ_2 in the complex plan, both of them lay the resolvent set of A,whichholds

$$\begin{aligned}\emptyset_j(k,hA) &= \frac{1}{2\pi i}\int_{\Gamma_1} \emptyset_j(k,\lambda)\,(\lambda I - hA)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \left(\frac{1}{2\pi i}\int_{\Gamma_2} e^{k\xi}\Psi_j(\xi,\lambda)d\xi\right)(\lambda I - hA)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_2} e^{k\xi}\left(\frac{1}{2\pi i}\int_{\Gamma_1} \Psi_j(\xi,\lambda)(\lambda I - hA)^{-1} d\lambda\right) d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_2} e^{k\xi}\Psi_j(\xi,hA)d\xi = L^{-1}[\Psi_j(.,hA)](k)\,.\end{aligned} \qquad (17)$$

4.2 Evaluation of the mapping by the Runge-Kutta methods

For φ_j in (9), $j \geq 1$and $t > 0$,the following formula is considered [8, 9,15]

$$\varphi_j(\lambda) = \int_0^1 e^{(1-\sigma)\lambda} \frac{\sigma^{j-1}}{(j-1)!}\, d\sigma = L^{-1}\left[L[g_0(.,\lambda)] \times L[g_j] \right](1),$$

where for $\sigma > 0$, $\lambda \in \mathbb{C}$,

$$g_0(\sigma,\lambda) = e^{\sigma\lambda} \text{and} g_j(\sigma) = \frac{\sigma^{j-1}}{(j-1)!}. \qquad (18)$$

For every $j \geq 1$ and $\in \mathbb{C}$, the following formula is defined

$$\Phi_j(z,\lambda) = L[g_0(.,\lambda)](z) \times L[g_j](z) = \frac{1}{z^j(z-\lambda)}, \tag{19}$$

and $\Phi_0(z,\lambda) = (z-\lambda)^{-1}$. Then, for every $\lambda \in \mathbb{C}$ and $j \geq 0$,

$$\varphi_j(\lambda) = L^{-1}[\Phi_j(.,\lambda)](1). \tag{20}$$

The same as argument in (17) justifies the operators $\varphi_j(hA)$ and $\varphi_j(c_l hA)$, $j \geq 0$, $2 \leq l \leq s$, which can be calculated by applying the inversion of the Laplace transforms

$$\Phi_j(z,\beta hA) = \frac{1}{z^j}(z-\beta hA)^{-1}, \quad j \geq 0, \quad \beta = 1, c_l, \tag{21}$$

to approximate the original mapping at $\sigma = 1$ [8].

5. Numerical illustration

In this part, the same examples are proposed [6, 16].

5.1 Example for the multistep exponential methods

The following problem is demonstrated [16]

$$u_t(x,t) = u_{xx}(x,t) + \left(\int_0^1 u(s,t)\,ds\right)u_x(x,t) + g(x,t), \tag{22}$$

for $x \in [0,1]$ and $t \in [0,1]$, subject to homogeneous Dirichlet boundary conditions and with $g(x,t)$ such that the exact solution to (22) is $u(x,t) = x(1-x)e^t$. Moreover, in the example, the mapping

$$\varphi(\lambda) = \frac{e^\lambda - 1}{\lambda}, \tag{23}$$

is considered [8].

$\lambda < 0$	$K = 15$	$K = 25$	$-\lambda$	$K = 15$	$K = 25$
-1	1.5050e-12	1.3323e-15	1	1.5050e-12	3.3307e-15
-1e-1	1.5227e-12	3.2196e-15	1e-1	1.5227e-12	3.5527e-15
-1e-2	1.4243e-12	4.4409e-15	1e-2	1.4243e-12	4.6629e-15
-1e-3	1.3750e-12	1.3323e-15	1e-3	1.3750e-12	1.3323e-15
-1e-4	1.3738e-12	1.7764e-15	1e-4	1.3738e-12	1.7764e-15
-1e-5	1.3747e-12	3.6637e-15	1e-5	1.3747e-12	3.7748e-15
-1e-6	1.3748e-12	3.6637e-15	1e-6	1.3748e-12	3.7748e-15
-1e-7	1.3695e-12	1.9984e-15	1e-7	1.3695e-12	1.9984e-15
-1e-8	1.3717e-12	1.1102e-16	1e-8	1.3717e-12	2.2204e-16
-1e-9	1.3715e-12	1.1102e-16	1e-9	1.3715e-12	0
-1e-10	1.3711e-12	0	1e-10	1.3711e-12	0
-1e-11	1.3711e-12	0	1e-11	1.3711e-12	0
-1e-12	1.3715e-12	1.1102e-16	1e-12	1.3715e-12	0
-1e-13	1.3712e-12	0	1e-13	1.3712e-12	2.2204e-16

Table 1: $\varphi(\lambda)$ in (23) for $\lambda \in [-1,1]$ is computed,and the absolute error obtained is displayed by MATLAB with $K = 15$ and $K = 25$.

In the table, the spatial discretization of (22) is implemented by applying standard finite differences with $J = 512$ spatial nodes, centered for the approximation of u_x.

The nonlocal term has been presented by the composite Simpson's formula [16]. Moreover, the formula (6) is utilised with $k = 1, 2, 3, 4$, for integrating in time the semi discrete problem so that A is $(J-1) \times (J-1)$ matrix

$$A = J^2\, tridiag\,([1, -2, 1]).$$

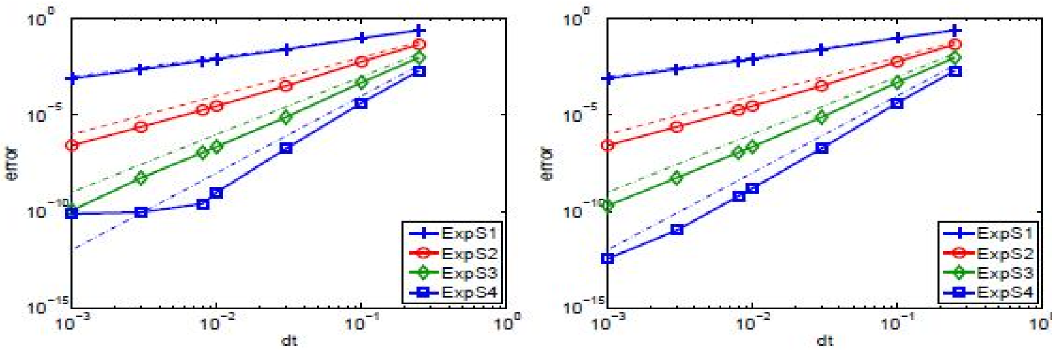

Figure 1: Error of exponential multistep methods (6) utilised to (22), for $k = 1, 2, 3$, and 4. Left: With $K = 25$ quadrature nodes on the hyperbolas, Right: With $K = 35$.

In Figure 1, the error versus the step size at $t - 1$ is shown and measured in a discrete version of the norm $\|.\|_{1/2}$, for $K = 25$ and $K = 35$. In Figure 1 also, lines of slope 1, 2, 3, and 4 are observed to visualise the order of convergence [16]. In fact, it is higher than the one predicted in [16].

5.2Example of the exponential Runge-Kutta methods

In this section, two examples are proposed [6]. The first following problem is considered [6]

$$u_t(x,t) = u_{xx}(x,t) + \frac{1}{1+u(x,t)^2} + g(x,t), \tag{24}$$

for $x \in [0,1]$ and $t \in [0,1]$, subject to homogeneous Dirichlet boundary conditions and with $g(x,t)$ such that the exact solution to (24) is $u(x,t) = x(1-x)e^t$. Moreover, the Butcher tableaus are applied with the abbreviations

$$\varphi_i = \varphi_i(hA), \quad \text{and } \varphi_{i,j} - \varphi_{i,j}(hA) - \varphi_i(c_j hA), \qquad 2 \le j \le s.$$

This problem is discretised in space by standard finite differences with $J = 200$grid points. For the time integration of semidiscrete problem, the equations (8) are considered with $s = 1$, and the second order method

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,2} & \\ \hline & 0 & \varphi_1 \end{array} \tag{25}$$

the third-order method

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3}\varphi_{1,2} & & \\ \frac{2}{3} & \frac{2}{3}\varphi_{1,3} - \frac{4}{3}\varphi_{2,3} & \frac{4}{3}\varphi_{2,3} & \\ \hline & \varphi_1 - \frac{3}{2}\varphi_2 & 0 & \frac{3}{2}\varphi_2 \end{array} \tag{26}$$

and the fourth-order method

$$\begin{array}{c|ccccc} 0 & & & & & \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,2} & & & & \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,3} - \varphi_{2,3} & \varphi_{2,3} & & & \\ 1 & \varphi_{1,4} - 2\varphi_{2,4} & \varphi_{2,4} & \varphi_{2,4} & & \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,5} - 2a_{5,2} - a_{5,4} & a_{5,2} & a_{5,2} & a_{5,4} & \\ \hline & \varphi_1 - 3\varphi_2 + 4\varphi_3 & 0 & 0 & -\varphi_2 + 4\varphi_3 & 4\varphi_2 - 8\varphi_3 \end{array} \tag{27}$$

with

$$a_{5,2} = \frac{1}{2}\varphi_{2,5} - \varphi_{3,4} + \frac{1}{4}\varphi_{2,4} - \frac{1}{2}\varphi_{3,5}$$

and

$$a_{5,4} = \frac{1}{4}\varphi_{2,5} - a_{5,2}.$$

In the implementation of the second order method, to invert four different Laplace transforms, the form of (21) is needed for approximating $\varphi_0(\frac{h}{2}A)$, $\varphi_0(hA)$, $\varphi_1(\frac{h}{2}A)$, and $\varphi_1(hA)$. In addition, the inversion of eight Laplace transforms is required in the implementation of the third-order and fourth-order methods.

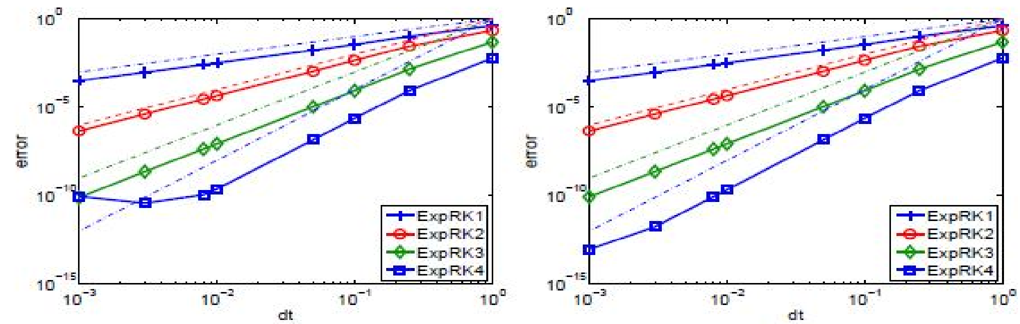

Figure 2: Error of Runge-Kutta methods (8) with $s = 1$, (25), (26), and (27) utilized to (24).
Left: With $K = 25$ quadrature nodes, Right: $K = 35$.

In Figure 2, the error at $t = 1$ versus the step size is displayed and measured in the maximum norm. In order to test the algorithm, lines with the corresponding slopes in Figure 2 are added. For this kind of methods, full precision for $K = 35$ are seen [6].

The second following problem is presented [6]

$$u_t(x,t) = u_{xx}(x,t) + \int_0^1 u(s,t)ds + g(x,t), \qquad (28)$$

for $x \in [0,1]$ and $t \in [0,1]$, subject to homogeneous Dirichlet boundary conditions and with $g(x,t)$ such that the exact solution to (28) is $u(x,t) = x(1-x)e^t$.

This problem in space is discretized as in the previous example (22), and the composite Simpson's rule is applied for the approximation of the nonlocal term. For the time integration, the formulas (8) are used with $s = 1$, (25), and (26).

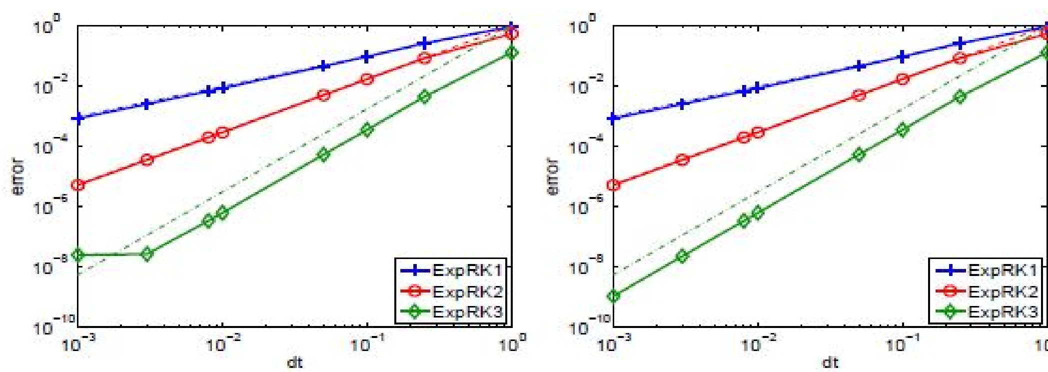

Figure 3: Error of Runge-Kutta methods (8) with $s = 1$, (25), and (26) applied to (28).
Left: With $K = 20$ quadrature nodes, Right: With $K = 25$.

In Figure 3, the error at $t = 1$ is shown and measured in a discrete version of the norm $\|.\|_{1/2}$ [6]. The order of convergence for this problem is expected, where the first order is 1, 1.75 for (25), and 2.75 for (26). Lines with the corresponding slopes in Figure 3 are added in order to check the algorithm for exponential methods. Full precision with only $K = 25$ qudrature nodes is achieved [6].

6. Conclusions

In this work, the way to approximate the exponential operators required for the implementation of different kinds of exponential methods has been derived, which have been demonstrated for the time integration of semi-linear problems. Moreover, the numerical inversion of the Laplace transform and its applications have been shown. Two examples of the multistep exponential and the exponential Runge-Kutta methods have also been considered.

REFERENCES

[1] H. Berland, B. Skaestad, and W. M. Wright. EXPINT - A Matlab Package for Exponential Integrators. ACM Transactions on Mathematical Software, 33:Article Number 4, 2007.

[2] S. M. Cox and P. C. Matthews, Exponential time differencing for stiff systems, J. Comput. Phys., 176, 430–455, 2002.

[3] A. K. Kassam. High Order Time stepping for Stiff Semi-Linear Partial Differential Equations. PhD thesis, Oxford University, 2004.

[4] A.-K. Kassam and L. N. Trefethen, Fourth-order time stepping for stiff PDEs, SIAM J. Sci. Comp., 26, 1214–1233, 2005.

[5] M. Hochbruck, A. Ostermann, *Exponential Runge–Kutta methods for parabolic problems*, Appl. Numer. Math. 53, Issues 2-4, 323-339, 2005.

[6] M. Hochbruck, A. Ostermann, *Explicit exponential Runge–Kutta methodsfor semi linear parabolic problems*, SIAM J. Numer. Anal. 43, No. 3, 1069-1090, 2005.

[7] D. Henry, Geometric theory of semi linear parabolic equations. Lecture Notes in Mathematics 840, Springer, Berlin, 1981.

[8] M. L´opez- Fern´andez, C. Palencia, A. Schädle, *A spectral order method for inverting sectorial Laplace transforms*, SIAM J. Numer. Anal., 44, pp.1332– 1350, 2006.

[9] M. L´opez- Fern´andez, C. Palencia, *On the numerical inversion of the Laplace transform of certain holomorphic mappings*, Appl. Numer. Math., 51,pp. 289–303, 2004.

[10] M. L´opez- Fern´andez, C. Lubich, C. Palencia, A. Schadle, *Fast Runge-Kutta approximation of inhomogeneous parabolic equations*, Numer. Math., 102, pp. 277–291, 2005.

[11] J. A. C. Weideman, L. N. Trefethen, *Parabolic and hyperbolic contours for computing the Bromwich integral*, Math. Comput. 76, pp. 1341–1356, 2007.

[12] I. P. Gavrilyuk, W. Hackbusch, B. N. Khoromskij, *Data-sparse approximation to a class of operator-valued functions*, Math. Comp., 74, pp. 681–708, 2005.

[13] D. Sheen, I. H. Sloan, V. Thom´ee, *A parallel method for time discretization of parabolic equations based on Laplace transformation and quadrature*, Math. Comp., 69, pp. 177–195, 2000.

[14] G. H. Golub and C. F. Van Loan. Matrix Computations. The Johns Hopkins University Press, Baltimore, MD, third edition, 1996.

[15] M. L´opez-Fern´andez, On the implementation of exponential methods for semi- linear parabolic equations, arXiv. math.NA, 2008.

[16] M. P. Calvo, C. Palencia, *A class of explicit multistep* exponential *integrators for semi-linear problems*, Numer. Math. 102, pp. 367–381, 2006.

[17] J. D. Lawson. Generalized Runge-Kutta Processes for Stable Systems with Large Lipschitz Constants. SIAM J. Numer Anal., 4:372-380, 1967.

[18] B. V. Minchev and W. M. Wright. A Review of Exponential Integrators for First Order Semi-Linear Problems. Tech. Rep. NTNU, Preprint. 2005.

[19] J. Certaine. The Solution of Ordinary Differential Equations with Large Time Constants. In Mathematical Methods for Digital Computers, A. Ralston and H. S. Wilf, eds.:128-132, Wiley, New York, 1960.

[20] A. Pazy. Semi-groups of Linear Operators and Applications to Partial Differential Equations. Springer, New York, 1983.

[21] D. Sheen, I. H. Sloan, V. Thom´ee, *A parallel method for time discretization of parabolic equations based on Laplace transformation and quadrature*, Math. Comp., 69 , pp. 177–195, 2000.

[22] L. F. Shampine and C. W. Gear. A User's View of Solving Stiff Ordinary Differential Equations. SIAM Review, 21:1-17, 1979.

[23] Q. Nie, Y.-T. Zhang and R. Zhao, Efficient semi-implicit schemes for stiff systems, Journal of Computational Physics, 214 , 521-537. 2006.

[24] G. Beylkin and J. M. Keiser, On the adaptive numerical solution of nonlinear partial differential equations in wavelet bases, J. Comput. Phys., 132 ,233–259, 1997.

[25] T.Y. Hou, J.S. Lowengrub and M.J. Shelley, Removing the stiffness from interfacial flows with surface tension, J. Comput. Phys., 114 ,312. 1994.